Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

palace on the Cascine and to a musical entertainment in La Pergola, and the Marquis di Pepoli, as syndic of Bologna, offered them the hospitality of that renowned city. From all classes, eminent statesmen and learned professors from all parts of Italy, and above all from Dr. Maestri, the chief of the Statistical Department, and to whom the congress owes so much for its success, the most courteous and hearty attentions were received.

I will conclude by observing that, though our mere professional pursuits only comprise a small part of the varied and most important topics to which I have alluded, the same theory of probabilities on which they are based is capable of application to many of the others. Knowledge always repays the toil by the pleasure itself of its acquisition. Still more must this be the case when the mind is given to subjects in which the health and happiness of thousands is involved and the progress of civilization advanced. If, therefore, when the members of this Institute have thoroughly grounded themselves in the knowledge of their profession, they should be disposed to extend the application of their science to some of the questions of which such a wide field of view is here opened up, they will have the satisfaction of knowing that in abolishing error, prejudice, or ignorance, they are using the basis of their professional skill for the good of their own community and the social progress of nations.

On the final law of the sums of drawings. *By* A. De Morgan, Esq.

LET there be letters $x, y, z, \ldots$ each of which has values, choices, or drawings. Let their number be σ, and let, $\xi, \eta, \zeta, \ldots$ be their several numbers of drawings: $x_1, x_2, \ldots x_\xi$ the drawings of x; and so on. Let $\Sigma x, \Sigma y, \ldots$ be the sums of the drawings of x, y, &c.; and $\Sigma{:}x$, $\Sigma{:}y$, the average drawings: so that $\Sigma{:}x = \frac{1}{\xi}\Sigma x$, and so on. Let any drawing of one letter be compatible with any drawings of the others: so that the terms of the product $\Sigma x.\Sigma y.\Sigma z$ contain every joint drawing of xyz. The number of drawings of xyz is $\xi\eta\zeta$: and $\Sigma(xyz) = \Sigma x.\Sigma y.\Sigma z$. Dividing both sides by $\xi\eta\zeta$, we have $\Sigma{:}(xyz) = \Sigma{:}x.\Sigma{:}y.\Sigma{:}z$. The same holds for any number of letters, or powers of letters: and thus we have the following theorem:—All possible combinations of drawings being taken into account, the average of all the

drawings of a product is the *similarly* resolved* product of the averages of all the drawings of the factors. The sum $x+y+z+\ldots$ has $\xi\eta\zeta\ldots$ drawings, say N. For a given set of drawings x_a, y_b, z_c, of a given number of letters, the expression $x_a+y_b+z_c+v+w\ldots+$ has $\mathrm{N}:\xi\eta\zeta$ drawings. Consequently the number of drawings of $(x+y+z+v+w+\ldots)^k$ in which $\mathrm{P}x_a^\alpha y_b^\beta z_c^\gamma$ occurs $(\alpha+\beta+\gamma=k)$ contributes $\mathrm{NP}x_a^\alpha y_b^\beta z_c^\gamma:\xi\eta\zeta$ to $\Sigma(x+y+z+\ldots)^k$, and, dividing by N, contributes $\mathrm{P}x_a^\alpha y_b^\beta z_c^\gamma:\xi\eta\zeta$ to the average. But this, by the preceding theorem, when a sum is made from all values of a, b, c, is P multiplied by the average of the product $x^\alpha y^\beta z^\gamma$. Hence this theorem;—The average of the drawings of $(x+y+z+\ldots)^k$ is the sum of the averages of the several terms in the multinomial development of $(x+y+z+\ldots)^k$.

Next, let h be the number of letters in a certain term; four, for instance, in $\mathrm{P}x^\alpha y^\beta z^\gamma v^\delta$. The whole number of letters being σ, the number of such terms, no two of which have the same letters, is h_σ, the number of combinations of h out of σ. If we take one such term, and if all the letters $\alpha, \beta, \gamma, \ldots$ be different, the number of terms containing these letters is the number of arrangements of h, or $h(h-1)\ldots 3.2.1$, say $[h$. But if there be repetitions, say for instance α occurs a times, β b times, &c., so that $\mathrm{a}\alpha+\mathrm{b}\beta+\mathrm{c}\gamma+\ldots=k$, $\mathrm{a}+\mathrm{b}+\mathrm{c}+\ldots=h$, then the number of terms in which one set of letters occurs is the number of distinct ways of distributing a marked α, b marked β, &c., in $\mathrm{a}+\mathrm{b}+\mathrm{c}+\ldots$ places: or $[h\div[\mathrm{a}.[\mathrm{b}.[\mathrm{c}\ldots.$ Hence the total number of terms of the type $\mathrm{P}x^\alpha y^\beta z^\gamma\ldots$ is h multiplied by the preceding or $[h.\sigma(\sigma-1)\ldots.(\sigma-h+1)$ divided by $[h.[\mathrm{a}.[\mathrm{b}.[\mathrm{c}\ldots.$ If σ increase without limit, this approaches to ratio of equality with $\sigma^h\div[\mathrm{a}.[\mathrm{b}.[\mathrm{c}\ldots$ and this we may write as the infinite number of terms of the type given, when the number of letters is infinite. In this case, the type having the greatest value of h gives a result infinitely greater than all the others put together, *if the drawings be all positive*, which at first we shall suppose. The degree of the term being uniform $(=k)$, h is greatest (and $=k$) when all the exponents α, β, &c., are severally equal to unity. The multinomial coefficient of such a term is $[k$. Hence, σ being infinite, $(x+y+z+\ldots)^k$ is $[k\times$ the sum of all products of $x, y, \ldots k$ and k together: or the rejected part is an infinitely small fraction of the retained part. Hence, by the preceding theorem, we find that

* Thus Σx^2 is not $\Sigma x.\Sigma x$, if ξ drawings of x^2 be found from ξ drawings of x. But Σxx *is* $\Sigma x.\Sigma x$, if ξ^2 drawings of xx be made by combining ξ drawings of the first x with ξ drawings of the second.

the average of $(x+y+\ldots)^k$ is [k × the sum of all products of $\Sigma:x$, $\Sigma:y$, &c. k and k together. But if we repeat this reasoning upon $(\Sigma:x+\Sigma:y+\ldots)^k$ we find the same [k × sum of combinations of $\Sigma:x$, $\Sigma:y$, &c. k and k together, for the value of this new multinomial power. Consequently

$$\Sigma:(x+y+\ldots)^k=(\Sigma:x+\Sigma:y+\ldots)^k$$

or:—All drawings being positive, and the number of letters infinite, the average drawing of the kth power of the sum is a *subequal** of the power of the sum of the several average drawings.

All the theorems which come under this subject have a remarkable quality. By the look of the demonstration, it should seem as if we must take a large number of letters to have a chance of even a glimpse of verification. It is not so: we may get a good glimpse out of three letters. Let each letter have drawings 1, 3; and take $x+y$. Its drawings of $x+y$ are 2, 4, 4, 6; the sum of averages is $2+2$. The averages of the sums of the 1st, 2nd, 3rd, 4th, powers of 2, 4, 4, 6 are 4, 18, 88, 456; of which it can only be said that the powers of $2+2$ are not forcibly suggested. Take three letters: the values of $x+y+z$ are 3, 5, 5, 7, 5, 7, 7, 9; and the average sums of powers 6, 39, 270, 1965; these much more nearly suggest the powers of $2+2+2$.

I now come to cases of both positive and negative drawing. But the drawings are to be *balanced:* that is, for every positive drawing the corresponding negative drawing exists; and *vice versâ.* Some very appropriately call such things *plusminus* drawings. Thus if 7, 7, 7 occur among the drawings of x, so do -7, -7, -7. Very slight attention will now make it apparent that in $\Sigma(x+y+\ldots)^k$ every term $Px^\alpha y^\beta\ldots$ in which one or more of $\alpha, \beta, \ldots$ are odd gives a set of drawings the sum of which vanishes. And as this must happen in *every* term when k is odd (since $\alpha+\beta+\ldots=k$) it follows that $\Sigma(x+y+\ldots)^k$ is always $=0$, when k is odd. We proceed to consider even powers, as in $(x+y+\ldots)^{2k}$. The reasoning of the preceding case, if understood, will lead us immediately to a new result. All cases being disposed of as evanescent in which one or more exponents are odd, the terms of the form $Px^2y^2\ldots$ having k letters in each, give a sum infinitely above that

* I use this word, which I find more and more convenient, to denote equality within an infinitely small part of the whole. Thus the circle is a subequal of the inscribed regular polygon with an infinite number of sides. No! surely the polygon is *sub*equal of the circle? Not so: the Latin preposition *sub*, thus used, does for both sides. *Subcruda* is used to mean short of raw on the cookery side, very little cooked; there is no underraw meat: *coquito paulisper uti subcruda fiet.* Thus the area of a curve and the sum of the inscribed rectangles ydx are subequal each of the other.

obtained from all the other terms. And P is $1.2.3\ldots 2k \div (1.2)^k$ or $1.3.5\ldots 2k-1 \times 1.2.3\ldots k$. Hence $\Sigma(x+y+\ldots)^{2k}$ is subequal of $1\ldots 2k-1 \times 1\ldots k \times$ the sum of all products of $x^2, y^2, z^2, \ldots$ k and k together. And in this last we obtain $\Sigma:(x+y+\ldots)^{2k}$ by writing $\Sigma:x^2$ for x^2, &c. But by the preceding theorem $1\ldots k$ $\times$ (the sum of all products of $\Sigma:x^2$, &c. k and k together) is subequal of $(\Sigma:x^2+\Sigma:y^2+\ldots)^k$, or (since $\Sigma:xy=0$) of $\{\Sigma:(x+y+\ldots)^2\}^k$. Hence, if A_{2k} represent the average $2k$th power of all the values of $x+y+\ldots$ we have, if each letter be of balanced drawings,

$$A_{2k}=1.3.5\ldots 2k-1A_2^k \quad . \quad . \quad . \quad . \quad (A)$$

In this and the preceding case the letters need not have the same set of values. Nevertheless, for a rough attempt at verification, let us suppose four letters, with drawings -1, 0, $+1$ for each. Then $x+y+z+v$ has 81 drawings; and if we write down each drawing as often as it occurs (counting negatives as positives, since we only want even powers) we have for the sum of the $2k$th powers of $x+y+z$

$$2.4^{2k}+8.3^{2k}+20.2^{2k}+32.1^{2k}+19.0^{2k}$$

giving 216, 1512, 15336, all divided by 81, or $\frac{8}{3}$, $\frac{56}{3}$, $\frac{568}{3}$, for attempts at A_2 A_4 A_6. Now $1.3A_2^2$ is $\frac{64}{3}$, not far from $\frac{56}{3}$; $1.3.5A_2^3$ is $\frac{7680}{27}$, instead of $\frac{5112}{27}$.

We now ask, what function of k, say ϕk, satisfies the functional equation (A). Without entering on the mode of finding, it will be enough here to state and verify that one solution is

$$A_{2k}=\sqrt{\frac{c}{\pi}}\int_{-\infty}^{\infty}\varepsilon^{-ct^2}t^{2k}dt.$$

It is a very common question to show, by integration by parts, that this gives

$$A_{2k}=\frac{2k-1}{2c}A_{2k-2}=\frac{2k-1}{2c}\cdot\frac{2k-3}{2c}A_{2k-4}=\ldots=\frac{2k-1}{2c}\cdots\frac{1}{2c}\cdot A_0.$$

And A_0 is known to be 1, whence A_2 is $\frac{1}{2c}$. Hence it appears that this form of A_{2k} gives $A_{2k}=(2k-1)\ldots 3.1.A_2^k$. The next question is, Are there any other solutions? Let the preceding be $\phi(2k)$, and let the most general value of A_{2k}, divided by $\phi(2k)$,

give $\psi(2k)$. We have then $A_{2k}=\phi(2k).\psi(2k)$; substitute this in (A) and strike out the equal factors $\phi 2k$ and $(2k-1)\ldots 1(\phi 2)^k$, which gives $\psi(2k)=(\psi 2)^k$, of which the only solution, so far as integers are concerned, is $\psi(2k)=m^{2k}$, m being any constant. The complete solution of (A) is then

$$A_{2k}=\sqrt{\frac{c}{\pi}}\cdot m^{2k}\int_{-\infty}^{+\infty}\varepsilon^{-ct^2}t^{2k}dt$$

where c and m are any constants.

We now come to another question. The number of letters which contribute their drawings to a sum being infinite (practically very many; or by the ascertained, but undemonstrated, goodness of the approximation, even a very moderate number) we have formed a law by which, as soon as the average *square* of the sum is known—which we must know to determine c—we determine the average $2k$th power of the sum. Our new question must be, Does this mode of proceeding implicitly contain a case of the problem? Have we found that, be the laws of the drawings what they may, the final law of the average $2k$th powers is the same as if those laws had all given way to some assignable law, deducible from the preceding formula. And the answer is affirmative, which may be shown as follows.

The factor m^{2k} may be thrown away. It merely indicates that if all the drawings be affected by a new factor, the average $2k$th power of the sum is affected by the $2k$th power of that factor. Now remember that if the drawings of a letter, a_1, a_2, &c., severally occur l_1, l_2, &c. times, the average $2k$th power of the drawings is $\Sigma la^{2k}:\Sigma l$, and* $l_w:\Sigma l$ is the multiplier for a_w^{2k} in that average power; and the sum of all the terms of the form $l_w:\Sigma l$ is unity. If then we find the factors λ_1, λ_2, &c. in $\lambda_1 a_1^{2k}+\ldots$ such that $\Sigma\lambda=1$, we have a representation of the average $2k$th power of the drawings, upon the supposition that a_1, a_2, &c. occur as possible drawings in numbers of times proportional to λ_1, λ_2, &c.; or, N being the whole number of drawings, $N\lambda_1$, $N\lambda_2$, &c. times.

It is well known that $\sqrt{\frac{c}{\pi}}\int_{-\infty}^{+\infty}\varepsilon^{-ct^2}dt=1$. Denote $\sqrt{\frac{c}{\pi}}\,\varepsilon^{-ct^2}$ by ft: then $\int_{-\infty}^{+\infty}ft.t^{2k}dt$ represents the average $2k$th power of a draw-

* The student who has had some glimpse of the problems in probability will wonder what I am at: I tell him I am keeping the theory of probability out of the way. Or rather;—When a barrister attempts to address the court without gown and wig, the judge says, Mr. ——, I can't *see* you! Suppose the theory of probability out of costume, and not privileged to be seen or heard. Perhaps I may make the application in another paper.

ing: the supposition being that every value of t occurs as a drawing $\mathrm{N}ftdt$ times, where N, infinitely great, is the whole number of drawings. I cannot enter fully on this point: it must be enough here to say that the whole interval from $-\infty$ to $+\infty$ is divided into intervals of dt, as in $\ldots -3dt,\ -2dt,\ -dt,\ 0,\ dt,\ 2dt,\ 3dt, \ldots$ and the drawing zdt occurs times enough to make its collection the fraction $f(zdt)dt$ of the whole number of drawings.

We now see that the average $2k$th power of $(x+y+\ldots)$ is that of a drawing of one letter on the condition that every drawing is possible, and that the (infinite) number of drawings between a and b is the fraction $\sqrt{\frac{c}{\pi}}\int_a^b \varepsilon^{-ct^2}dt$ or $\int_a^b ft.dt$, of the infinite total number of drawings. And all we know of c is that $1:2c$ is the average square of a drawing, or $\int_{-\infty}^{+\infty} ft.t^2dt$.

If we take each one of the letters, $x, y, z, \ldots$ and take the average squares of their drawings separately, say $\frac{1}{2\alpha}$, $\frac{1}{2\beta}$, &c., and then make $\frac{1}{2\alpha}+\frac{1}{2\beta}+\ldots=\frac{1}{2c}$, we have the following result. Let λ be one of the letters $\alpha, \beta, \ldots$, and for the law of drawings of x substitute that the fraction of drawings between a and b, for any values of a and b, is $\int_a^b f(t,\lambda)dt$; where $f(t,\lambda)$ is $\sqrt{\frac{\lambda}{\pi}}\varepsilon^{-\lambda t^2}$. Do the same with all the letters. Then, the sum has for its law of drawing that the fraction of its drawings which lies between a and b, is $\int_a^b f(t,c)dt$. And the average $2k$th powers, for an infinite number of drawings, are accurately represented—and for a moderate number very nearly—by the several cases of $\int_{-\infty}^{+\infty} f(t,\lambda)t^{2k}dt$. And $\int_{-\infty}^{+\infty} f(t,c)t^{2k}dt$ represents the average $2k$th power of the sum.

One case of the theory of probabilities is the application of the preceding to *balanced* errors of observation, in which positive and negative errors are equally likely. It is shown that everything depends upon average even powers of errors: whence from the preceding it is made to follow that, be the law of error what it may, the results, on a moderate number of observations, are the same as if it were that the chance of an error between a and b is $\sqrt{\frac{c}{\pi}}\int_a^b \varepsilon^{-ct^2}dt$, where $1:2c$ is the average square of an error.

There are various grounds on which this law of error lies under suspicion of being very near to a physical truth, by the nature of men and things: but the last theorem renders this of small importance. The proof of this last theorem, as given by Laplace, demands the highest resources of the integral calculus. Such a megatherium as

$$\frac{1}{\pi}\int_0^\pi \cos l\varpi\, d\varpi \left[\psi\left(\frac{0}{n}\right)+\psi\left(\frac{1}{n}\right)\cdot 2\cos\varpi + \ldots \psi\left(\frac{n}{n}\right)\cdot 2\cos n\varpi\right]^s$$

must not be exhibited in an elementary museum.

A few years ago (*Camb. Trans.*, vol. x., part 2, Nov. 11, 1861) I succeeded in reducing the connexion of all laws with the *final law* to the comparatively easy form shown in this paper. The considerations introduced are such as may become useful in actuaries' work: and certainly will, if the higher mathematics continue to be applied in life contingencies as they have been of late years.

Difficulty may present itself to those who have no sufficient command of an integral as the sum of an infinitely great number of infinitely small elements. I shall not, for instance, be quite clear throughout this paper to a person on whom it does not flash, when I state it, that I have proved the following theorem:—If $\int_a^b \phi x dx$ be distributed into an infinite number of elements of the form $\phi x dx$, and if every combination of k elements be multiplied together, the sum of all the products is $\left(\int_a^b \phi x dx\right)^k \div 1.2\ldots k$. Required a very simple proof of this.

An instance of verification is always valuable; and I therefore give one case of each leading theorem, using the abbreviations of the *calculus of operations*.

Let each of the σ letters $x, y, \ldots$ give the drawings 0 and 1. The sum of the kth powers of all drawings of $x+y+\ldots$ is $0^k+1_\sigma 1^k+2_\sigma 2^k+\ldots+\sigma_\sigma \sigma^k$. Let E represent the operation of changing m into $m+1$: the above is the operation $(1+\mathrm{E})^\sigma$ performed upon 0 in 0^k. This is $(2+\Delta)^\sigma.0^k$; and since $\Delta^n 0^k=0$ when $n>k$, the highest term is $k_\sigma 2^{\sigma-k}\Delta^k 0^k$, which, since σ is infinite and $\Delta^k 0^k=1.2\ldots k$, is $\sigma^k 2^{\sigma-k}$. Divide by 2^σ, the number of drawings of $x+y+\ldots$, and we have $(\frac{1}{2}+\frac{1}{2}+\ldots\ \sigma \text{ terms})^k$, or the kth power of the sum of the averages.

Again, let each of the σ letters have the drawings -1 and $+1$. The sum of the $2k$th powers of the drawings of $x+y+\ldots$ is $\sigma^{2k}+1_\sigma(\sigma-2)^{2k}+2_\sigma(\sigma-4)^{2k}+\ldots+\sigma_\sigma(-\sigma)^{2k}$ which is $(\mathrm{E}+\mathrm{E}^{-1})^\sigma$

applied to 0 in 0^{2k}. This is $\{2+\Delta^2(1+\Delta)^{-1}\}^\sigma 0^{2k}$; and its highest term has $\Delta^{2k}(1+\Delta)^{-k}0^{2k}$, in the development of which only $\Delta^{2k}0^{2k}$ has value. The term to be retained is therefore $k_\sigma 2^{\sigma-k}\Delta^{2k}0^{2k}$, or, σ being infinite,

$$\sigma^k 2^{\sigma-k}\Delta^{2k}0^{2k} \div 1.2\ldots k, \text{ or } \sigma^k 2^{\sigma-k}\times 1.3.5\ldots 2k-1\times 2^k,$$

$\Delta^{2k}0^{2k}$ being $1.2.3\ldots 2k$. Divide by 2^σ, the number of drawings of $x+y+\ldots$ and we have $\sigma^k\times 1.3\ldots 2k-1$, in which σ, or $(1+1+1+\ldots \sigma$ terms), is the sum of the average squares of the drawings of $x, y, \ldots$

The theorem which contains the signification of $\sigma(\sigma-1)\ldots(\sigma-a-b-c-\ldots+1)$ divided by $[a.[b.[c\ldots$ seldom or never finds a place in chapters on combination. I recommend it to the attention of the elementary student, who may find various demonstrations of it.

On the Rate of Interest in Loans repayable by Instalments. By PETER GRAY, F.R.A.S., *Honorary Member of the Institute of Actuaries.*

(Continued from page 102.)

MY second example is the Austrian Loan of 1865. The conditions are as follows:—

734,694 Bonds, each of £19. 17*s*. 0*d*., issued at £13. 14*s*. 4*d*., that is, at a discount of £6. 2*s*. 8*d*. each.

£1	0	0	per Bond to be paid on application, say Dec. 1, 1865, and the remaining £12. 14*s*. 4*d*. in the following instalments:—
1	19	7	on Dec. 15, 1865,
3	11	7	,, Feb. 10, 1866,
3	11	7	,, April 10, ,, ,
3	11	7	,, June 10, ,, ,
13	14	4	Total.

Subscribers will be at liberty to pay their Scrip in full on any one of the above dates, under discount at 6 per cent per annum.

The bonds are to bear interest at the rate of 9*s*. 11*d*. each (=·4958333) per half year, (a trifle under $2\frac{1}{2}$ per cent on the *nominal* amount) payment of which to become due on the 1st June and the 1st Dec. of each year.

The bonds are to be redeemed in thirty-seven years, 9928 (to be selected by lot) half-yearly, at the same dates as the payments of